Generating *k*-Pell Infinite Series Using Whittaker's Formula

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Abstract

In this paper, we apply Whittaker's formula on 2nd degree polynomial equations of the form: $x^2 + 2x - (k-1)$, where $k \geq 2$ and $k \in N$. By applying Whittaker's formula on these polynomial equations, we obtain infinite series for $\sqrt{k}$ that involve integers belonging to specific k-Pell sequences. More specifically, the infinite series of $\sqrt{k}$ will be made of terms containing the successive integers belonging to the k-Pell sequence $\{P_{k-1,n}\}_n$.

Whittaker's formula is a method that can be used on polynomials to obtain an infinite series that converges to the root that has the smallest absolute value. The formula is very elegant and it probably deserves more attention than it received. In this paper we will apply Whittaker's formula on a specific family of polynomials in order to obtain infinite series for square roots of positive integers greater than 1. The infinite series obtained are interesting because the terms of the series involve integers belonging to *k*-Pell sequences. The most known *k*-Pell sequence is the Pell sequence or the Pell numbers. It was known since antiquity that you can use Pell numbers to obtain a rational approximation of $\sqrt{2}$. A rational approximation of square roots of integers can be obtained using continued fractions. The infinite series discussed in this paper can be seen as alternatives to continued fractions.

k-Pell Sequences

k-Pell sequences are sequences of integers that are defined by recursive recurrences. For $n \geq 1$, the *k*-Pell sequence $\{P_{k,n}\}_n$ is defined by the following recurrence relation and initial conditions: $P_{k,n} = 2\,P_{k,n-1} + k\,P_{k,n-2}$, $P_{k,0} = 0$ and $P_{k,1} = 1$. When $k = 1$, the sequence is the original Pell sequence $\{P_n\}_n$ that gives the name to the entire family of related sequences. Using the following formula and initial conditions $P_n = 2\,P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$, we get the following sequence: 0,1,2,5,12,29,70,169, … (This is sequence A000129 in the OEIS database). The first few values of each k-Pell sequence can be easily obtained using the corresponding recurrence relation. The first 4 or 5 consecutive values of each k-Pell sequence can be typed in the OEIS (The Online Encyclopedia of Integer Sequences) search engine to find the sequence in the OEIS database. The OEIS database provides a list of interesting properties for each sequence.

Whittaker Formula

Whittaker's formula is a method that uses the coefficients of the polynomial equation to create some special matrices. The determinants of these special matrices are used to create an infinite series that converges to the root that has the smallest absolute value. If we have the following general polynomial $0 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$, the smallest root in absolute value is given by the equation:

$$-\frac{a_0}{a_1} - \frac{a_0^{\,2} a_2}{a_1 \det\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^{\,3} \det\begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\det\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \det\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} - \frac{a_0^{\,4} \det\begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\det\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} \det\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} - \cdots$$

The formula above was discovered by E.T. Whittaker one hundred years ago, so for convenience we will call it Whittaker's formula. The Whittaker formula has a few advantages over other numerical methods for finding the roots of polynomial equations. Other numerical methods, like Newton's method, require to approximate the location of the root, calculate the derivative of the polynomial at specific locations and do iterations until you get an accurate result. On the other hand, the Whittaker formula only requires the use of the coefficients in an infinite series formula. Unfortunately, the Whittaker formula has a disadvantage. The Whittaker formula cannot be used if there are two roots with the smallest absolute value. For example, the quadratic equation x^2-k has the roots $\sqrt{k}$ and $-\sqrt{k}$. Since abs($\sqrt{k}$) = abs ($-\sqrt{k}$), the Whittaker formula will not converge. To get a greater understanding of Whittaker formula, I recommend the textbook coauthored by Whittaker himself listed at [1] (free on archive.org).

Polynomial Transformation

We want to obtain infinite series for square root of positive integers k by applying the Whittaker formula. Unfortunately, we cannot use the formula on the quadratic equations of the form x^2-k since the formula will not converge. To remedy the problem, we can use the following polynomial transformation that replaces x by (x+1): $(x + 1)^2 - k = x^2 + 2x + 1 - k = x^2 + 2x - (k-1)$, where $k \geq 2$ and $k \in N$. This new polynomial has the roots $\sqrt{k} - 1$ and $-\sqrt{k} - 1$. For integers $k \geq 2$, abs($\sqrt{k} - 1$) < abs($-\sqrt{k} - 1$). Thus, if we apply the Whittaker formula to this new 2nd degree polynomial, the infinite series should converge to $\sqrt{k} - 1$.

Our general polynomial $x^2 + 2x - (k-1)$ has the following assigned coefficients: $a_0 = -(k-1)$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$ etc. Thus

$$\sqrt{k} - 1 = -\frac{-(k-1)}{2} - \frac{[-(k-1)]^2 1}{2\det\begin{vmatrix} 2 & 1 \\ -(k-1) & 2 \end{vmatrix}} - \frac{[-(k-1)]^3 \det\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}{\det\begin{vmatrix} 2 & 1 \\ -(k-1) & 2 \end{vmatrix} \det\begin{vmatrix} 2 & 1 & 0 \\ -(k-1) & 2 & 1 \\ 0 & -(k-1) & 2 \end{vmatrix}}$$

$$- \frac{[-(k-1)]^4 \det\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -(k-1) & 2 & 1 \end{vmatrix}}{\det\begin{vmatrix} 2 & 1 & 0 \\ -(k-1) & 2 & 1 \\ 0 & -(k-1) & 2 \end{vmatrix} \det\begin{vmatrix} 2 & 1 & 0 & 0 \\ -(k-1) & 2 & 1 & 0 \\ 0 & -(k-1) & 2 & 1 \\ 0 & 0 & -(k-1) & 2 \end{vmatrix}} - \cdots$$

Simplifying the infinite series

The matrices in the numerator of the terms of the infinite series have the following form

$$M_{n\times n} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -(k-1) & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -(k-1) & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -(k-1) & 2 & 1 \end{vmatrix}$$

We can see that $M_{n\times n}$ is a lower triangular number. The determinant of a triangular matrix is obtained by multiplying the elements on the diagonal. In our case det $M_{n\times n}$ = $(1)^n$= 1. Thus, the terms of our infinite series are greatly simplified.

On the other hand, the matrices found in the denominator have the following form

$$P_n(k-1)=\begin{vmatrix} 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ -(k-1) & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & -(k-1) & 2 & \dots & 0 & 0 & 0 \\ 0 & 0 & -(k-1) & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 2 & 1 & 0 \\ 0 & 0 & 0 & \dots & -(k-1) & 2 & 1 \\ 0 & 0 & 0 & \dots & 0 & -(k-1) & 2 \end{vmatrix}$$

$P_n(k-1)$ is a special tridiagonal matrix. Using methods discussed in the paper [2], we can compute the determinants for the first few n the following way:

$$\det P_1(k-1)=2$$

$$\det P_2(k-1)=2 \det P_1(k-1) - [-(k-1)](1)=4 + (k-1)$$

$$\det P_3(k-1))= 2 \det P_2(k-1) +(k-1) \det P_1(k-1)$$

$$\det P_4(k-1)= 2 \det P_3(k-1) +(k-1) \det P_2(k-1)$$

and, in general

$$\det P_n(k-1)= 2 \det P_{n-1}(k-1) +(k-1) \det P_{n-2}(k-1)$$

We can see that the general form looks like a k-Pell recurrence relation. Using the table from paper [2], we see that det $P_1(k-1)$=2 = $P_{k-1,2}$. Similarly, det $P_2(k-1)$= 4 + (k-1)= $P_{k-1,3}$. Thus, from the initial values and from the general form, we deduce that the $n\times n$ matrix $P_n(k-1)$ has the determinant det $P_n(k-1)$= $P_{k-1,n+1}$.

The infinite series can be written using the following general form

$$\sqrt{k}-1= -\frac{-(k-1)}{2} - \frac{[-(k-1)]^2}{2\times det\ P_2(k-1)} - \frac{[-(k-1)]^3}{detP_2(k-1)\times detP_3(k-1)} - \frac{[-(k-1)]^4}{detP_3(k-1)\times detP_4(k-1)} - \dots$$

and using the proper substitution we get

$$\sqrt{k}-1= -\frac{-(k-1)}{2} - \frac{[-(k-1)]^2}{2\times P_{k-1,3}} - \frac{[-(k-1)]^3}{P_{k-1,3}\times P_{k-1,4}} - \frac{[-(k-1)]^4}{P_{k-1,4}\times P_{k-1,5}} - \dots$$

A concrete example and a more elegant general formula

Until now we dealt with more general forms and formulas. The smallest integer for which we can obtain an infinite series using the method presented in this paper is k=2. Our corresponding polynomial is $x^2 + 2x - (2-1) = x^2 + 2x - 1$. Thus, we can assign the following coefficients $a_0 = -1$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$ etc. In this case the infinite series will be associated with the sequence $\{P_{1,n}\}_n$ or $\{P_n\}_n$. Finally, we obtain the following initial infinite series

$$\sqrt{2}-1 = -\frac{-1}{2} - \frac{(-1)^2 1}{2\det\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}} - \frac{(-1)^3 \det\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}{\det\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}\det\begin{vmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}} - \frac{(-1)^4 \det\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{vmatrix}}{\det\begin{vmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}\det\begin{vmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{vmatrix}} - \dots$$

By applying simplifications obtained from the previous sections we get

$$\sqrt{2}-1 = -\frac{-1}{2} - \frac{(-1)^2}{2\times 5} - \frac{(-1)^3}{5\times 12} - \frac{(-1)^4}{12\times 29} - \dots$$

Finally, we may deduce the following more general formula

$$\sqrt{2}-1 = \frac{1}{1\times 2} - \frac{1}{2\times 5} + \frac{1}{5\times 12} - \frac{1}{12\times 29} + \dots + \frac{(-1)^{n+1}}{P_n \times P_{n+1}}$$

The first term in the last infinite series was modified to allow to write a more general formula. Using the last formula, we can obtain a more elegant general formula for $k \geq 2$

$$\sqrt{k}-1 = \frac{(k-1)}{1\times 2} - \frac{(k-1)^2}{2\times P_{k-1,3}} + \frac{(k-1)^3}{P_{k-1,3}\times P_{k-1,4}} - \frac{(k-1)^4}{P_{k-1,4}\times P_{k-1,5}} + \dots + \frac{(-1)^{n+1}(k-1)^n}{P_{k-1,n}\times P_{k-1,n+1}}$$

Final Remarks

As we mentioned in the introduction, the infinite series obtained with the Whittaker formula can be considered as alternatives to infinite continued fractions. In this paper we applied the Whittaker formula on 2nd degree polynomials that have a specific form. To obtain different infinite series for the same $\sqrt{k}$, the readers can use a different polynomial transformation before they apply the Whittaker formula. For example, we can replace x by (x+2) to obtain $(x+2)^2 - k = x^2 + 4x - (k-4)$. The polynomials that result from different polynomial transformations will probably give infinite series that don't involve *k*-Pell integers when the Whittaker formula is applied. Nonetheless, the infinite series will probably have terms belonging to other interesting integer sequences. Also the readers can compare the convergence rate for different transformations to see which one is more efficient.

We also encourage the readers to apply the Whittaker formula on various unrelated polynomials. For example, the readers should try the Whittaker formula on the polynomial $x^2 + x - 1$. The root with the smallest absolute value in this case is equal to the golden ratio conjugate (~0.61803). The readers should try to see a pattern in the terms given by the Whittaker formula.

References

[1] Whittaker E.T. and Robinson G., The Calculus of Observations, pp 120-123, 1924

[2] Paula Catarino, "On Generating Matrices of the k-Pell, k-Pell-Lucas and Modified k-Pell Sequences", Pure Mathematical Sciences, 3 (2), 71-77, 2014