Lill’s Method and the Sum of Arctangents

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Abstract

In this paper we will show that the polynomial equations with real roots have a special property when they are represented using the graphic method called Lill's method. This special property will allow us to construct graphically the sum of arctangents.

Introduction

Lill's method is a remarkable visual method that can be used to represent and solve graphically polynomial equations with real coefficients. In this paper, we will concentrate on polynomial equations that have only real roots. We will show that the polynomial equations with real roots have a useful property when they are graphed using Lill's method. This property will allow us to obtain graphically the sum of arctangents by constructing a corresponding polynomial using Lill's method.

Lill's method

Let's have the general polynomial of order n, $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, where the coefficients $\{a_n, a_{n-1}, \ldots, a_1, a_0\}$ are real numbers. In Lill's method, these $n+1$ coefficients are represented by $n+1$ "reflective" and perpendicular segments that are connected in a link or path, and each segment will have its length equal to the absolute value of the corresponding coefficient. The path starts with the segment P_0P_1 corresponding to a_n, then the segment corresponding to a_{n-1} or P_1P_2, is constructed such that P_1P_2 is perpendicular to P_0P_1. The process is repeated until the segment corresponding to a_0 is constructed. Each of these n "reflective" segments must also be extended (at infinity) in both directions by "refractive" or dotted lines. Also each of these segments must have a direction, and I will use the following formula for length and direction of a segment: $a_ke^{i(n-k)\pi/2}$, where a_k is the coefficient corresponding to the segment, i is the imaginary number and $0\leq k \leq n$.

In order to understand Lill's representation of a polynomial, we can graph the polynomial $P(x)= x^4 + 2x^3 +3x^2 + x +1$, with $a_4=1$, $a_3=2$, $a_2=3$, $a_1=1$, $a_1=1$. The segment corresponding to the coefficient $a_4=1$ is P_0P_1. Using the direction formula we get $1e^{i(4-4)\pi/2}=e^0=1$. So, if we consider P_0 to be the point of origin, P_1 is one unit to the right of P_0. The segment corresponding to the coefficient $a_3=2$ is P_1P_2, with the direction $2e^{i(4-3)\pi/2}=2e^{i\pi/2}=2i$. Therefore, P_2 is 2 units up from P_1. Now P_2 becomes the starting point of the segment corresponding to a_2. The direction formula gives -3 for the segment P_2P_3 corresponding to a_2. We also get -i for P_3P_4 corresponding to $a_1=1$ and 1 for P_4P_5 corresponding to $a_0=1$. The graphical representation of the polynomial can be seen in Figure 1, where P_0 is the origins of the coordinate system. Looking at Figure 1, we can make a few additional observations. If the coefficients a_k and a_{k-1} have the same sign, the segment corresponding to a_{k-1} is 90 degrees counterclockwise in relation to the segment corresponding to a_k. If the coefficients a_k and a_{k-1} have different signs, the segment corresponding to a_{k-1} is 90 degrees clockwise in relation to the segment corresponding to a_k. A more important observation for this paper is that when the coefficients a_k and a_{k-2} have the same sign, the segments corresponding to the a_k and a_{k-2} go in opposite directions. For example, in Figure 1 the segment corresponding to $a_3=2$ is P_1P_2, which goes up since P_1 is the starting point of the segment. The segment corresponding to $a_1=1$ is P_3P_4, which goes down since P_3 is the starting point. The same can be said about the segments corresponding to a_4 and a_2, or a_2 and a_0.

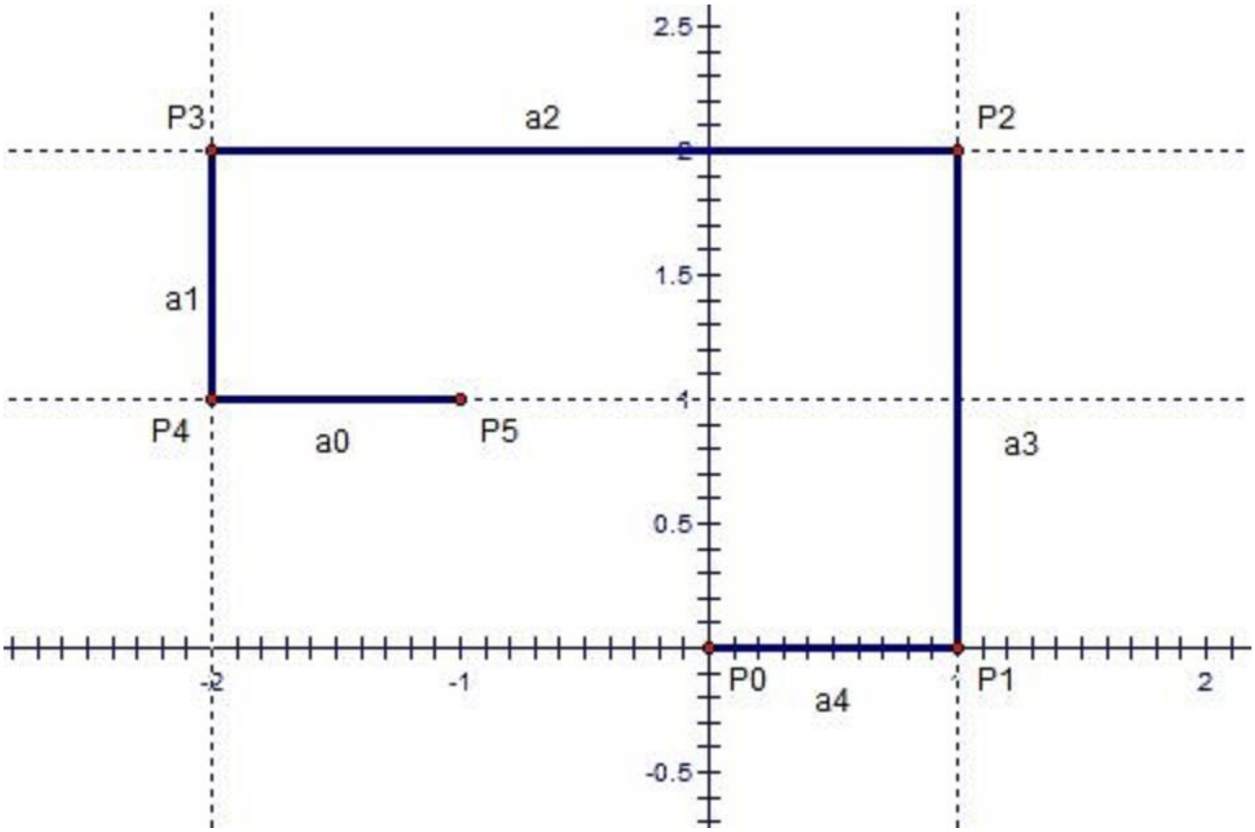

Figure 1

In order to obtain a solution for a Lill representation, we must find a path made of n perpendicular segments that starts at P_0 and ends at P_{n+1}. For example, let's have the polynomial $P(x)=x^2+2x+1$, with $a_2=1$, $a_1=2$ and $a_0=1$. The root of this polynomial is $x_r=-1$, with the multiplicity 2. Figure 2 displays the solution. The solution path is created by the segment P_0A_1 and the segment A_1P_3. The general rule is to start from P_0 by creating a segment P_0A_1 that intersects the segment corresponding to the coefficient a_{n-1} or the dotted line extension of a_{n-1}. The segment P_0A_1 makes an angle θ with the segment corresponding to a_n or P_0P_1. From A_1 you create another segment that makes an angle θ with the segment corresponding to a_{n-1}, and the new segment should intersect the segment corresponding to the coefficient a_{n-2} or its extension. The process is repeated until you get the segment $A_{n-1}P_{n+1}$. In Figure 2 we can see that 2 successive segments from the solution path are always perpendicular. Thus, the segment A_1P_3 is perpendicular to the segment P_0A_1. The numerical solution is given by the formula $x_r = -\tan(\theta) = A_1P_1/P_0P_1$. In our example $x_r = -1$ and $\theta=45$ degrees. So, the angle θ is positive if it is counterclockwise with respect to P_0P_1, and negative if it is clockwise.

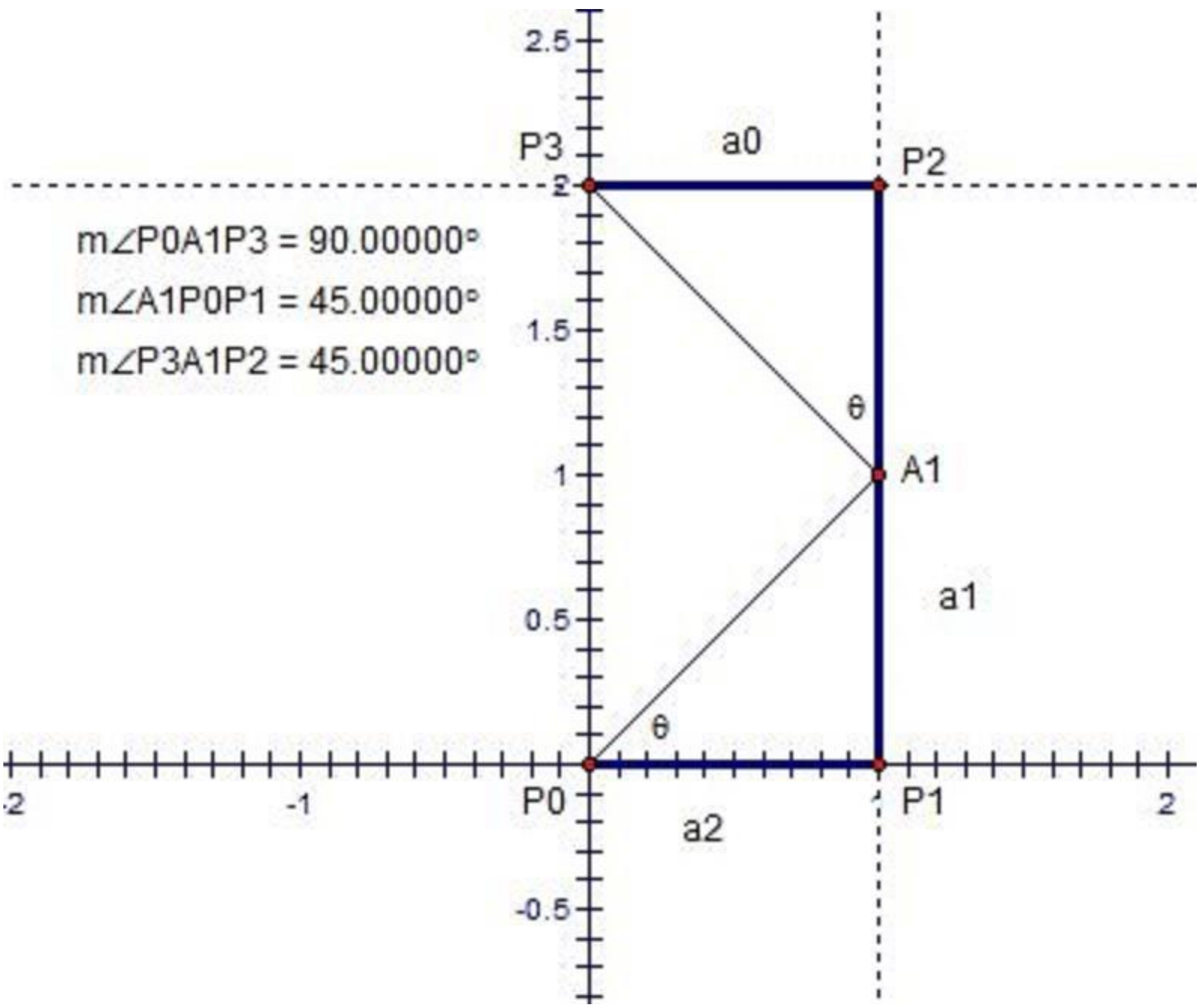

Figure 2

To get a better grasp of Lill's method, let's consider the polynomial $P(x)= x^2-x-1$, with $a_2=1$, $a_1=-1$ and $a_0=-1$. One of the solutions is the golden number $x_r=\varphi\sim 1.618$. Figure 3 shows the solution path. Figure 2 shows a solution path that involved "reflection", while Figure 3 shows a solution path that involves "refraction". We see that the segment P_0A_1 intersects the dotted extension of P_1P_2. In Figure 2, the segments P_0A_1 and A_1P_3 are on the left side of P_1P_2. In figure

3 we see that the segment A_1P_3 crossed to the right side of P_1P_2. If a solution segment intersects a dotted extension, the next solution segment always goes beyond the dotted extension. We should also observe that the angle $\theta = m(P_1P_0A_1)$ should be negative since it is clockwise with respects to P_0P_1. So $\theta \sim -58.28253$. In Figure 3, we again observe that 2 consecutive segments that belong to the solution path are perpendicular since A_1P_3 is perpendicular to the segment P_0A_1. The last observation we can make is the fact that we ignored the intersection of the segment P_0A_1 with the dotted extension of P_2P_3. If a solution segment goes from the segment corresponding to a_k (or its dotted extension) to the segment corresponding to a_{k-1} (or its dotted extension), the segments corresponding to other coefficients (and their dotted extensions) can be ignored.

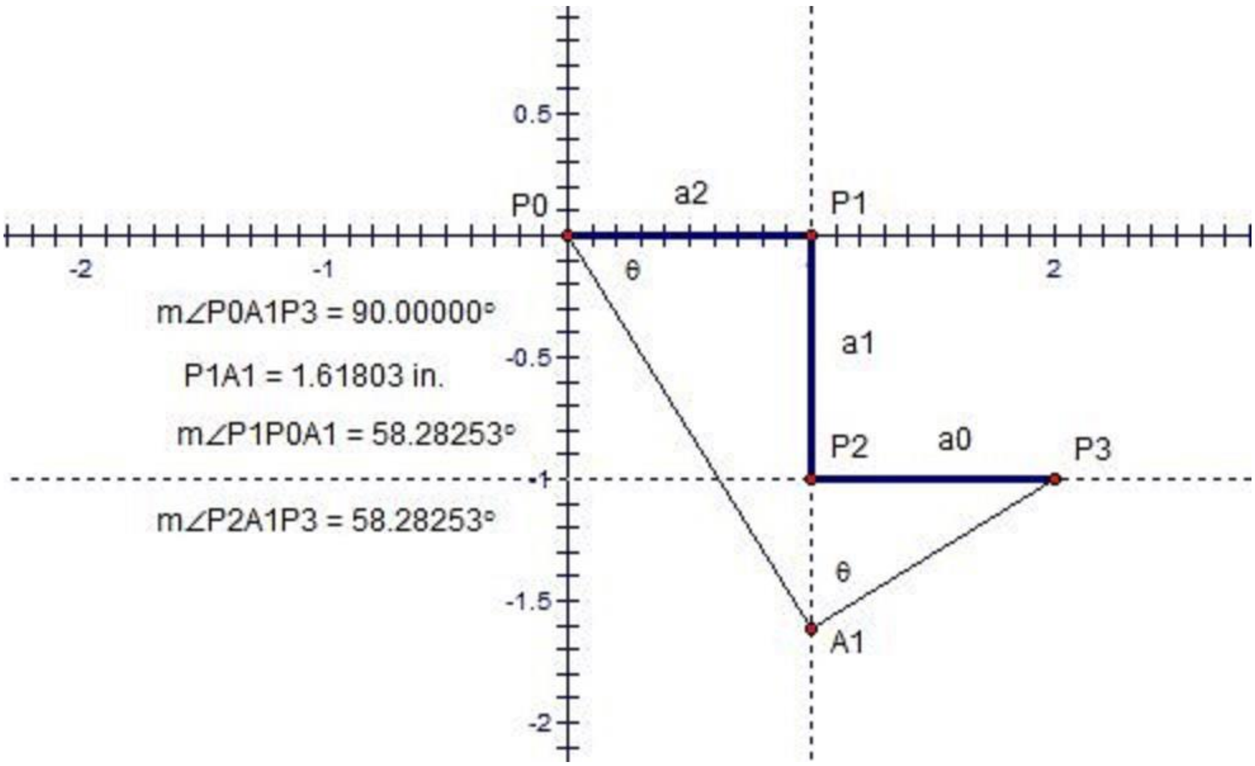

Figure 3

Vieta's Formulas

Vieta's formulas are very useful since they relate the coefficients of a polynomial to sums and products of its roots. Any general polynomial of order n, $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, has n roots $x_1,x_2,\ldots,x_n$. Vieta's formulas give the following relations:

$$x_1 + x_2 + \ldots + x_{n-1} + x_n = -\frac{a_{n-1}}{a_n}$$

$$(x_1x_2 + x_1x_3 + \ldots + x_1x_n) + (x_2x_3 + x_2x_4 + \ldots + x_2x_n) + \ldots + x_{n-1}x_n = \frac{a_{n-2}}{a_n}$$

$$\vdots$$

$x_1x_2\ldots x_{n-1}x_n = (-1)^n \frac{a_0}{a_n}$

We can replace a root x_k by $-\tan(\theta_k)$, where θ_k is the angle in a Lill diagram that gives a solution. However, x_k must be a real solution in order to have a corresponding angle θ_k such that $x_k = -\tan(\theta_k)$. If x_k is complex, Lill's solution is not given by a corresponding $-\tan(\theta_k)$ anymore. In the next sections, we will focus on polynomials that have only real roots.

Lill's Tangent of Sums Property

Let's have a general polynomial of order n, $P(x)=x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, that has only the real roots $x_1,x_2,...,x_n$. Each of these n roots has a corresponding angle $\theta_1,\theta_2,\ldots,\theta_n$ such that $x_1=-\tan(\theta_1)$, $x_2=-\tan(\theta_2),\ldots$, $x_n=-\tan(\theta_n)$. We also let the segment $P_{n-k}P_{n-k+1}$ be the Lill representation of the coefficient a_k. Then

$$\tan(P_1P_0P_{n+1}) = \tan(\theta_1 + \theta_2 + \ldots \theta_n)$$

Proof: We should remember that P_1P_0 is the segment corresponding to a_n, which is the first segment we draw using Lill's method. P_{n+1} is the terminus of the segment corresponding to a_0, and it is the last point drawn in a Lill representation of a polynomial. We also used the formula $a_k e^{i(n-k)\pi/2}$ to give the direction to the segment corresponding to a_k, which is $P_{n-k}P_{n-k+1}$. The formula gives a real number when $P_{n-k}P_{n-k+1}$ is horizontal, and a complex number when $P_{n-k}P_{n-k+1}$ is vertical. The tangent formula is $\frac{rise}{run}$, so we will add the vertical coefficient in the numerator and the horizontal coefficients in the denominator. The direction formula cannot be used directly since the numerator would be complex and the denominator would be real. Instead we can use the observation made in a previous section where we stated that when the coefficients a_k and a_{k-2} have the same sign, the segments corresponding to the a_k and a_{k-2} go in opposite directions. Using the observations mentioned we can write the following:

$$\tan(P_1P_0P_{n+1}) = \frac{a_{n-1}-a_{n-3}+a_{n-5}-\cdots}{a_n-a_{n-2}+a_{n-4}-\cdots} = \frac{a_{n-1}-a_{n-3}+a_{n-5}-\cdots}{1-a_{n-2}+a_{n-4}-\cdots}$$

This tangent equation can be compared to an equivalent tangent of sums equation from this paper [1]. But to make things a little bit simpler we can look at a 2^{nd} degree polynomial $P(x)=x^2+a_1x+a_0$, with the real roots x_1 and x_2. Then

$$\tan(P_1P_0P_3) = \frac{a_1}{1-a_0}$$

$$\text{by Vieta's formulas } = \frac{-[-\tan(\theta 1)-\tan(\theta 2)]}{1-[-\tan(\theta 1)][-\tan(\theta 2)]}$$

$$= \frac{\tan(\theta 1)+\tan(\theta 2)]}{1-[\tan(\theta 1)\tan(\theta 2)]}$$

$$= \tan(\theta 1 + \theta 2)$$

We can point out that we chose $a_n=1$ for convenience. If a_n is not equal to 1, we can always divide the polynomial by a_n to make the coefficient of x^n equal 1, and the new polynomial would have the same exact roots. We can also see that in Vieta's formulas, a_n divides the other coefficients. Thus, our formulas would yield the same results since a_n would be a common factor to all the terms in our equation.

Applications

A corollary of the tangent of sums property is the sum of angles property, which is $m(P_1P_0P_{n+1}) = \theta_1 + \theta_2 + \ldots \theta_n$. Since we deal with angles between -90 degrees and 90 degrees, maybe we can find an application that deals with sum of arctangents. For example, we can use this Lill property to construct graphically an angle that is the sum of 2 or more arctangents.

One well know identity is arctan(1) + arctan(2) + arctan(3)=180 degrees. We know that if $\theta=\arctan(x)$, then $x=\tan(\theta)$. To construct the sum of arctangents graphically, we can make a corresponding polynomial that has the roots $x_k=\tan(\theta_k)$. In Lill's method the roots are $-\tan(\theta)$, but we can deal with that later. So using our specific example we can have the polynomial with roots $x_1=1$, $x_2=2$ and $x_3=3$. We can use Vieta's formula to obtain the polynomial or we can expand $(x-1)(x-2)(x-3)= x^3-6x^2+11x-6$. So $a_3=1$, $a_2=-6$,$a_1=11$ and $a_0=-6$. Figure 4 shows the Lill representation of this polynomial. In Lill's method the solution angles are $\theta_1=-45$, $\theta_2\approx -63.43$ and $\theta_3\approx -71.56$. Thus $m(P_1P_0P_4)=-180$. We obtained the negative sign because we let $x_k=\tan(\theta_k)$ instead of $x_k=-\tan(\theta_k)$. We could have used $x_1=-1$, $x_2=-2$ and $x_3=-3$ to obtain positive angles.

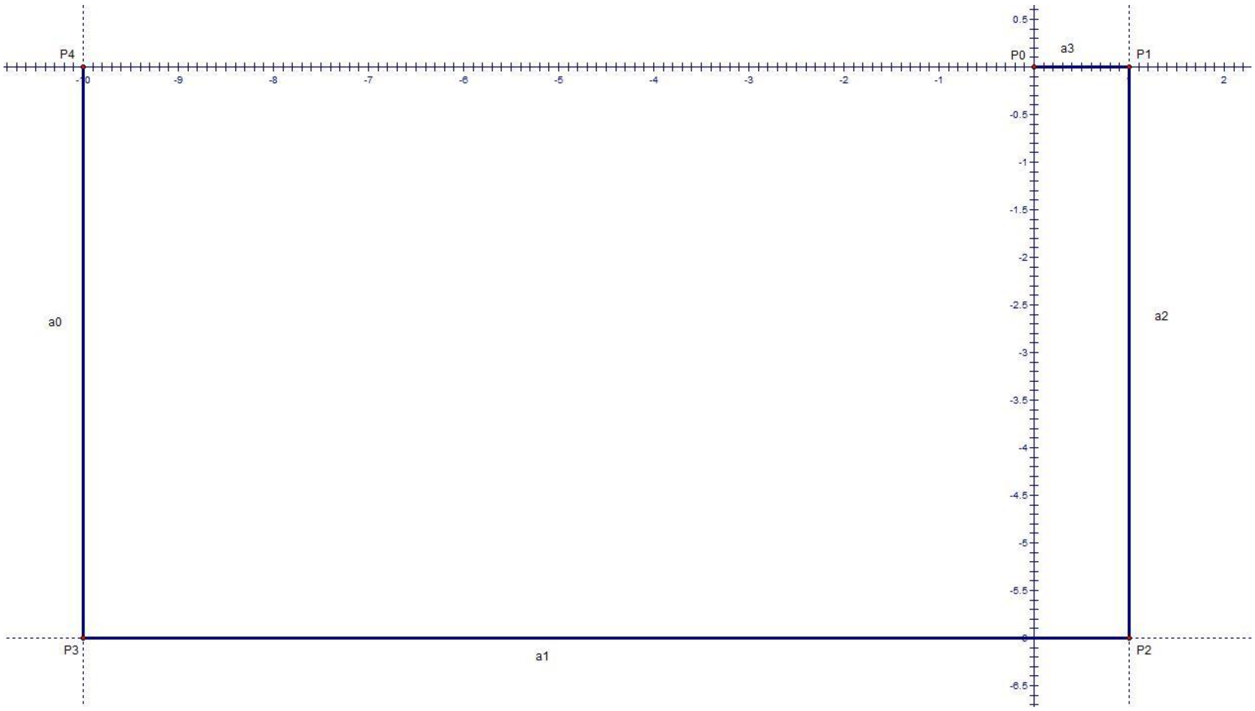

Figure 4

This method is probably best used when you have to add 2 or 3 arctangents. The corresponding polynomial becomes more complex as the sum has more terms. This method can be also adapted to find the tangent of sum of angles, if the tangent of individual angles is known.

Final Notes

Lill's method has many additional application that were not covered in this paper. For example, you can construct certain circles in order to obtain the solutions to quadratic equations, even if the solutions are complex. To learn more about Lill's method we recommend papers [2] and [3].

The sum of angle property is another interesting aspect of Lill's diagrams since it shows how various branches of mathematics are connected. It shows how trigonometric identities like tangent of sums are connected to polynomial equations that have real roots. The sum of angles property seems to hold even when the polynomials have complex roots. Proving that the sum of angles property holds when the polynomial has complex roots (or complex and real roots) can be the subject of another paper.

References

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