Representing Polynomials with Complex Coefficients using Lill's Method

Abstract

In this paper, we will show how to represent polynomial equations with complex coefficients using Lill's method. We will also discuss various general properties.

Introduction

Lill's method is a remarkable visual method that can be used to represent and solve graphically polynomial equations. Lill's method was developed in 19^{th} century by the Austrian engineer Eduard Lill. All the papers that discussed Lill's method since the time of Lill himself, only dealt with polynomial equations with real coefficients. Eduard Lill showed in his second paper [1] how to represent the roots that have imaginary parts, but nobody discussed how to represent the polynomials that have coefficients with imaginary parts. By showing how to represent polynomials with complex coefficients, we make Lill's method more complete. At the end, we will also discuss some general properties and possible applications.

Graphing Polynomials

Let's have the general polynomial of order n, $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, where the coefficients $\{a_n, a_{n-1}, \ldots, a_1, a_0\}$ are complex numbers. In Lill's method the polynomial P(z) can be represented by a link of connected straight segments P_0 P_1 $P_2 \ldots$ P_n P_{n+1}, where P_kP_{k+1} is a straight segment corresponding to a_{n-k}, where $0\leq k \leq n$. But to construct the link we must know what length and direction to apply to the segments that make up the link. The segment P_{n-k} P_{n-k+1} that corresponds to the coefficient a_k is obtained with the following formula: $Re(a_k)e^{i(n-k)\pi/2} + Im(a_k)e^{i(n-k+1)\pi/2}$, where $0\leq k \leq n$ and i is the imaginary number.

To make the things more concrete, we can represent the 2^{nd} degree polynomial $P(z)=x^2+x+(1-i)$. Thus we have $a_2=1$, $a_1=1$ and $a_0=1-i$. The segment P_0 P_1 corresponding to $a_2=1$ is given by $Re(1)e^{i(2-2)\pi/2} + Im(1)e^{i(2-2+1)\pi/2}=e^0+0=1$. Thus, if we let P_0 to be the point of origins of our link, then P_1 is one unit to the right of P_0. Now P_1 is the starting point for the segment P_1 P_2 that corresponds to $a_1=1$. Similarly the segment P_1 P_2 is given by $Re(1)e^{i(2-1)\pi/2} + Im(1)e^{i(2-1+1)\pi/2}= e^{i\pi/2}+0=i$.Thus P_2 is one unit up with respect to P_1. Finally, we obtain the segment P_2 P_3 that corresponds to the coefficient $a_0=1-i$ and is given by $Re(1-i)e^{i(2-0)\pi/2} + Im(1-i)e^{i(2-0+1)\pi/2}=e^{i\pi}-e^{3i\pi/2}=-1-(-i)=-1+i$. Thus, P_3 is one unit to the left and one unit up with respect to P_2. Lill's representation of the polynomial P(z) is shown in Figure 1.

Before we go forward discussing how to obtain the roots of polynomials, we can make a few observations. If the successive coefficients a_k and a_{k-1} are real numbers, then the segments corresponding to them will be perpendicular. In Figure 1 we can see that P_1P_2 is perpendicular to P_0P_1. In papers [2] and [3] some extension of the Lill's method for representing polynomials with real coefficients was discussed, where the segments corresponding to a_k and a_{k-1} don't have to be perpendicular. However even this extended Lill method graphs a polynomial using a fixed angle φ, so the angle between the segments corresponding to a_k and a_{k-1} is 180-φ or φ. By using the method presented in this paper, we can see that the angle between the segments corresponding to a_k and a_{k-1} can be any angle between 0 and 180 degrees and it doesn't depend on any fixed angle. In Figure 1, we can see that P_1P_2 is perpendicular to P_0P_1, but P_2P_3 is at 135 degrees with respect to P_1P_2. Also, the extended Lill's method discussed in papers [2] and [3] can only represent polynomials with real coefficients. Nonetheless the 2 papers were an inspiration for this paper.

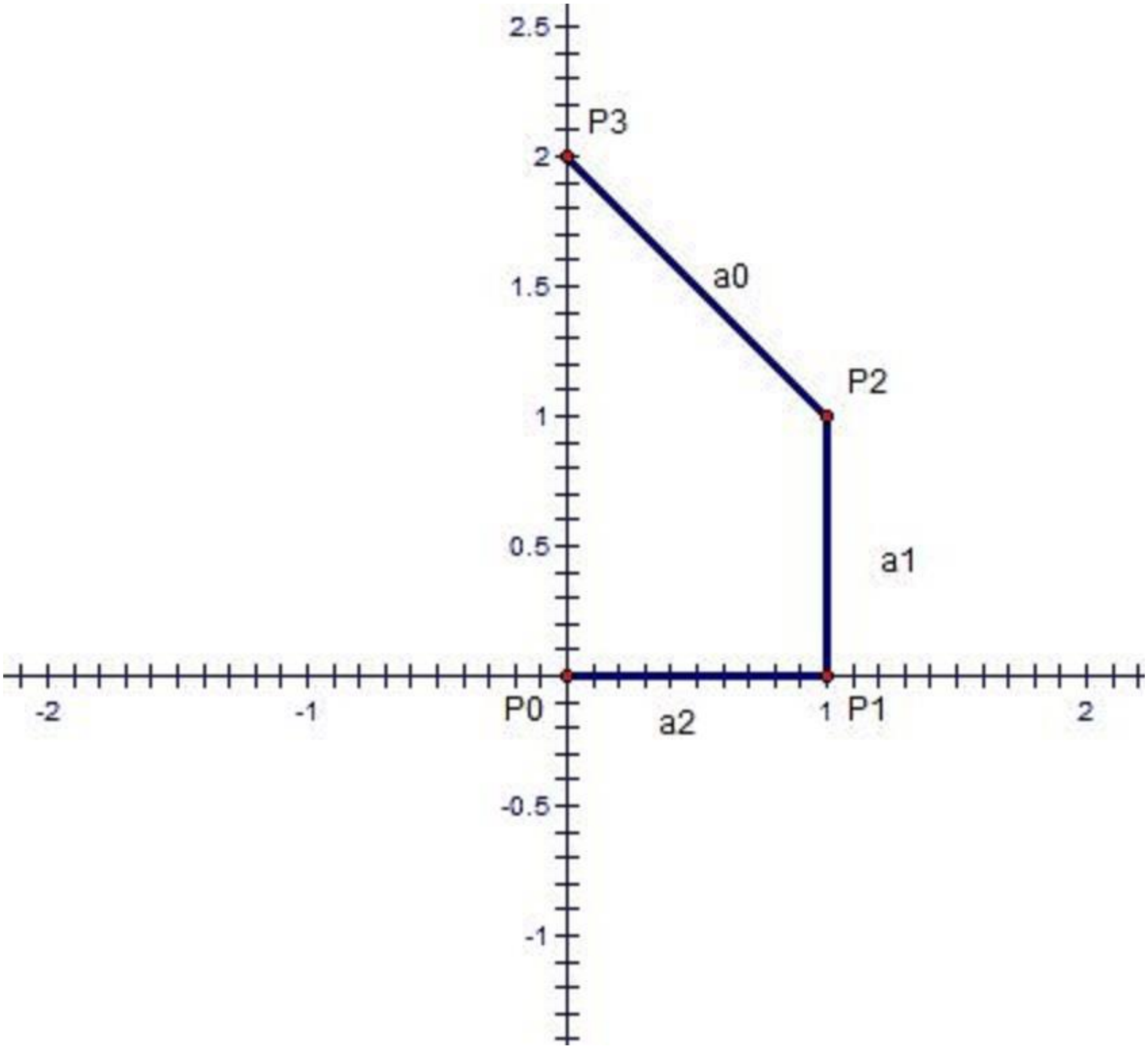


Figure 1

Solutions

A solution path to a general polynomial of order n, $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, where the coefficients $\{a_n, a_{n-1}, \ldots , a_1, a_0\}$ are complex numbers, is given by a path $P_0A_1A_2\ldots A_{n-1}P_{n+1}$ such that the triangles $P_0P_1A_1$, $A_1P_2A_2$, $A_2P_3A_3$, …, $A_{n-2}P_{n-1}A_{n-1}$ and $A_{n-1}P_nP_{n+1}$ are similar, and $m(P_1P_0A_1)=m(P_2A_1A_2)=\ldots=m(P_{n-1}A_{n-2}A_{n-1})=m(P_nA_{n-1}P_{n+1})=\theta$. The angle θ is the solution angle and it has a positive value when P_0A_1 is counterclockwise with respect to P_0P_1.Some special cases occur when the polynomial P(z) has roots with Re(z)=0. The most special case is z= -i, since θ is undefined for this value of z. Otherwise, if Re(z)=0 and z is not -i, θ can be 0 degrees or 180 degrees.

In the general case where z is not necessarily a root of the polynomial P(z), the path obtained using Lill's method is given by $P_0A_1A_2\ldots A_{n-1}A_n$, such that the triangles $P_0P_1A_1$, $A_1P_2A_2$, $A_2P_3A_3$, …, $A_{n-2}P_{n-1}A_{n-1}$, $A_{n-1}P_nA_n$ are similar and $m(P_1P_0A_1)=m(P_2A_1A_2)=\ldots=m(P_nA_{n-1}A_n)=\theta$. Thus, z is a root only when A_n is the same point as P_{n+1}. We can use some formulas from paper [4], to obtain the points $A_1,A_2,\ldots$ and A_n. Thus, taking the segment P_kP_{k+1} corresponding to a_{n-k} as our reference, we get:

$$A_kP_k \text{ represents } a_nz^k+\ldots+a_{n-k+1}z \quad (1)$$

and

$$A_kP_{k+1} \text{ represents } a_nz^k+\ldots+a_{n-k+1}z+a_{n-k} \quad (2)$$

So we obtain this useful equations:

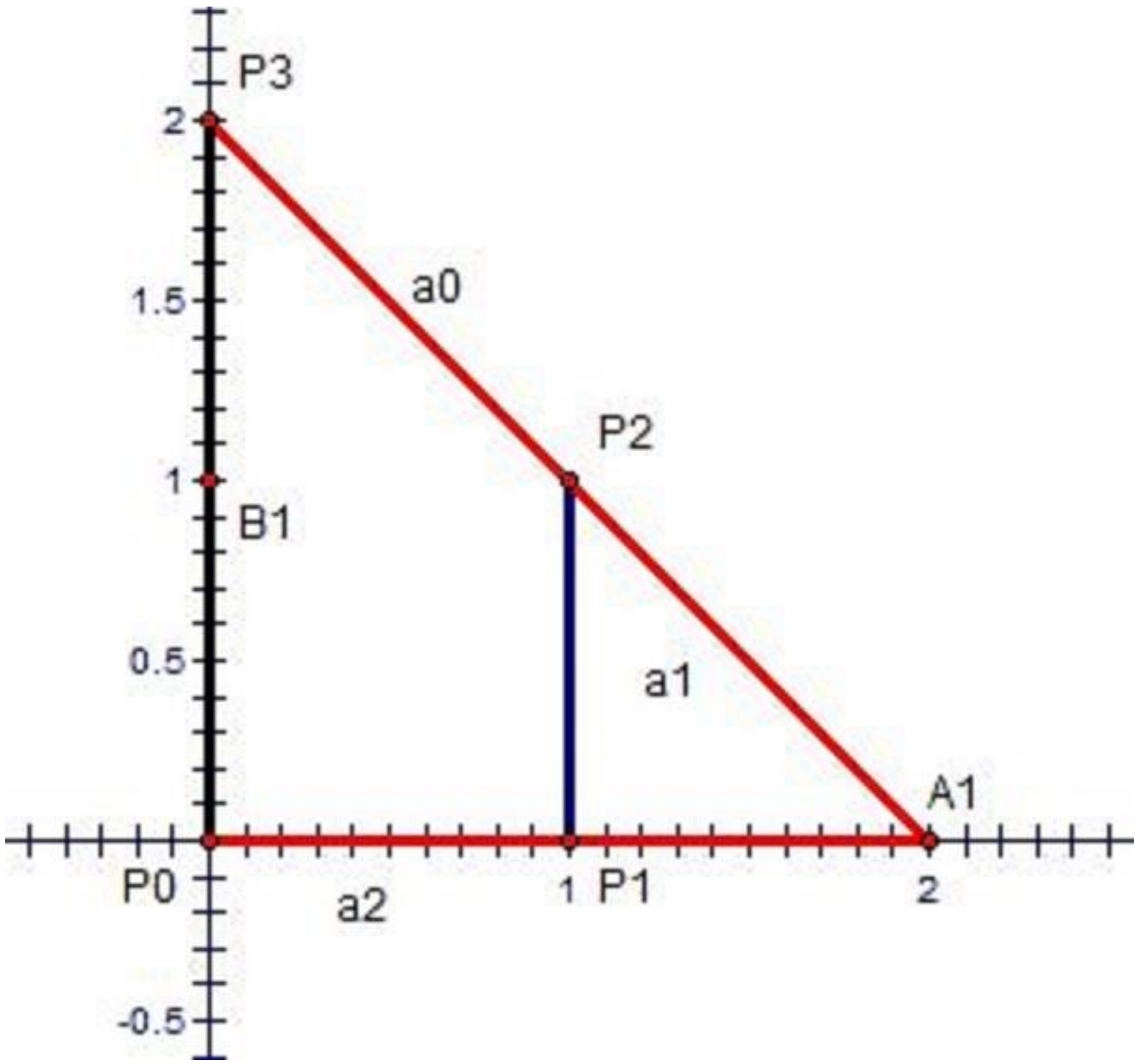
$$A_1P_1 \text{ represents } a_nz, \text{ with respect to } P_1P_2 \quad (3)$$

and

$$A_nP_{n+1} \text{ represents } a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0=P(z), \text{ with respect to } P_nP_{n+1} \quad (4)$$

Thus, when the points A_n and P_{n+1} are the same, we get P(z)=0 and z is a root.

To make things more concrete, we can graph the solution path of our polynomial $P(z)=x^2+x+(1-i)$. P(z) can be factored as (z-i)(z+(1+i)), so $z_1=i$ and $z_2=-1-i$. Figure 2 shows the path for both roots. Thus, the path for $z_1=i$ is given by $P_0A_1P_3$ and the path for $z_2==-1-i$ is given by $P_0B_1P_3$. We can also see that $m(P_1P_0A_1)=m(P_2A_1P_3)=0$, thus $\theta_1=0$. We already mentioned that the case when θ=0 is a special case since the triangle $P_0P_1A_1$ is actually a straight line because $m(P_1P_0A_1)=0$. On the other hand $m(P_1P_0B_1)=m(P_2B_1P_3)=90$. We can also see easily that the triangles $P_0P_1B_1$ and $B_1P_2P_3$ are similar triangles. We can also check using the equations (1)-(4). For example, $A_1P_1=-1$,so A_1P_1 goes 1 unit to left. If a_1 had a +i component, then the direction of the imaginary part would be $Im(i)e^{i(2-1+1)\pi/2}=(1)e^{i\pi}=(1)(-1)=(-1)$. Thus, if a_1 had a +i component, it would have the same direction as A_1P_1. We can say that A_1P_1 represents i with respect to the side P_1P_2. $B_1P_1= 1 - i$, and represents $-1 -i$ with respect to P_1P_2. The real part of B_1P_1 is opposite to $A_1P_1=-1$, so it should represent $-i$ with respect to P_1P_2. The imaginary part or the vertical part of B_1P_1 goes in the opposite direction of $P_1P_2=i$. Since P_1P_2 represents the coefficient $a_1=1$, the vertical component of B_1P_1 should represent -1 with respect to P_1P_2.



P3
2
a0
1.5
P2
1
B1
0.5
a1
A1
P0
a2
1
P1
2
-0.5

References

1) M. E. Lill (1868). "Résolution graphique des équations algébriques qui ont des racines imaginaires". Nouvelles Annales de Mathématiques. 2. 7: 363–367.

2) MEULENBELD, B., "*Note on LILL'S method of solution of numerical equations*". Proc. Kon. Ned Akad. V. Wetensch. 53, 464 (1950).

3) Dan Kalman, and Mark Verdi. "*Polynomials with Closed Lill Paths*." Mathematics Magazine, vol. 88, no. 1, 2015, pp. 3–10. JSTOR, www.jstor.org/stable/10.4169/math.mag.88.1.3.

4) Meulenbeld, B "NOTE ON THE REPRESENTATION OF THE VALUES OF POLYNOMIALS WITH REAL COEFICIENTS FOR COMPLEX VALUES OF THE VARIABLE"