Fibonacci Infinite Series and the Negative Powers of the Golden Ratio

Raul Prisacariu

http://raulprisacariu.com/

Introduction

In this short paper I will present 2 general infinite series formulas that show an interesting relationship between the Fibonacci numbers and the negative powers of the golden ratio (φ). I will also describe the method that I used to obtain the 2 general formulas. This paper is more informal, so I will ty to be brief. I am not sure if the 2 general formulas were already discovered. If the formulas are already known, they were probably obtained using a method different from the one presented in this paper.

General Formulas

The Fibonacci numbers are defined by the following recursive relation: F(k) = F(k-1) + F(k-2) with F(0) = 0 and F(1) = 1. Then the 2 general formulas for the negative powers of the golden ration are:

for k = odd natural number

$$\varphi^{-k}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}F(k)^2}{F(kn)F(kn+k)}$$

for k= even natural number

$$\varphi^{-k}=\sum_{n=1}^{\infty}\frac{F(k)^2}{F(kn)F(kn+k)}$$

Some Examples

Now that we know the general formulas, we can go over some examples.

$$0.618033989 \approx \varphi^{-1} = \frac{1}{(1)(1)} - \frac{1}{(1)(2)} + \frac{1}{(2)(3)} - \frac{1}{(3)(5)} + \frac{1}{(5)(8)} \dots + \frac{(-1)^{n+1}}{F(n)F(n+1)}$$

$$0.381966011 \approx \varphi^{-2} = \frac{1}{(1)(3)} + \frac{1}{(3)(8)} + \frac{1}{(8)(21)} + \frac{1}{(21)(55)} + \frac{1}{(55)(144)} \dots + \frac{1}{F(2n)F(2n+2)}$$

$$0.236067977 \approx \varphi^{-3} = \frac{2^2}{(2)(8)} - \frac{2^2}{(8)(34)} + \frac{2^2}{(34)(144)} - \frac{2^2}{(144)(610)} + \frac{2^2}{(610)(2584)} \dots + \frac{(-1)^{n+1}2^2}{F(3n)F(3n+3)}$$

$$0.055728090 \approx \varphi^{-6} = \frac{8^2}{(8)(144)} + \frac{8^2}{(144)(2584)} + \frac{8^2}{(2584)(46368)} + \frac{8^2}{(46368)(832040)} + \frac{8^2}{(832040)(14930352)} \dots + \frac{8^2}{F(6n)F(6n+6)}$$

The numerators of the terms from the infinite series have the square of F(1)=1, F(2)=1, F(3)=2 and F(6)=8. People can try to see if the formulas work for other negative powers of the golden ration.

Method

I want to go briefly over the method I used to obtain the 2 general formulas. To obtain the formulas I used Whittaker's formula for polynomial equations. Whittaker's formula can be used to find the smallest root of a polynomial (the root with the smallest absolute value). For more information about the Whittaker formula you can read [1] or my paper "Generating k-Pell Infinite Series Using the Whittaker Formula". To obtain the 2 infinite series formula I applied the Whittaker Formula on the polynomials $O_k(x)= -x^2 – L(k)\ x +1$, for k= odd natural number, and $E_k(x)= x^2 – L(k)\ x + 1$, for k= even natural number. L(k) refers to the kth Lucas number. The recursive relation for Lucas numbers is: L(n)= L(n-1) + L(n-2), where L(0)=2 and L(1)=1. Both $O_k(x)$ and $E_k(x)$ have the roots $-\varphi^k$ and φ^{-k}. Thus, you can use the Whittaker formula to obtain φ^{-k}, since it is the root with the smallest absolute value.

For k=1, we have the polynomial $O_1(x)= -x^2- x +1$. For k=2, we have the polynomial $E_2(x)= x^2 – 3x =2$. For k=3, we have $O_3(x)= -x^2 – 4x + 1$. For k=6, we have $E_6(x)= x^2 -18x +1$.

Final Notes

The 2 general infinite series formulas show again how closely connected are the Fibonacci numbers and the golden ratio. The polynomials discussed above involve Lucas numbers, which are also known to be connected to the Fibonacci numbers and the golden ratio. The 2 formulas and the polynomials discussed above show another interesting connection between the Lucas numbers, the Fibonacci numbers and the golden ratio.

I also want to encourage my readers to read more about the Whittaker formula. The formula can be used to obtain interesting infinite series.

References

[1] Whittaker E.T. and Robinson G., The Calculus of Observations, pp 120-123, 1924