The Pentagon, a Cyclic Quadrilateral, and a Geometric Approximation for the Cubic Root of 20 and Euler's Number

Raul Prisacariu

raulprisacariu.com

Introduction

In this paper I will show how to use a regular pentagon to obtain an interesting quadrilateral. I will star with a regular pentagon that has the sides length equal to 1 unit. Then using a few geometric constructions techniques, we will obtain a quadrilateral that has a few interesting properties.

The initial impetuous for discovering the geometric constructions presented in this paper came from my study of regular polynomials using Lill's method. Lill's method is an alternative graphic method of representing polynomial equations. In Lill's method, a polygon can be a graphical representation of a polynomial equation. I will add a few notes on Lill's method later in this paper. But this paper is more about a few geometrical curiosities, so I will not discuss polynomials or Lill's method too much.

Geometric Constructions

I will start with a regular pentagon ABCDE that has the sides equal to 1 unit as seen in Image 1. The readers can search online for a method of constructing such a pentagon.

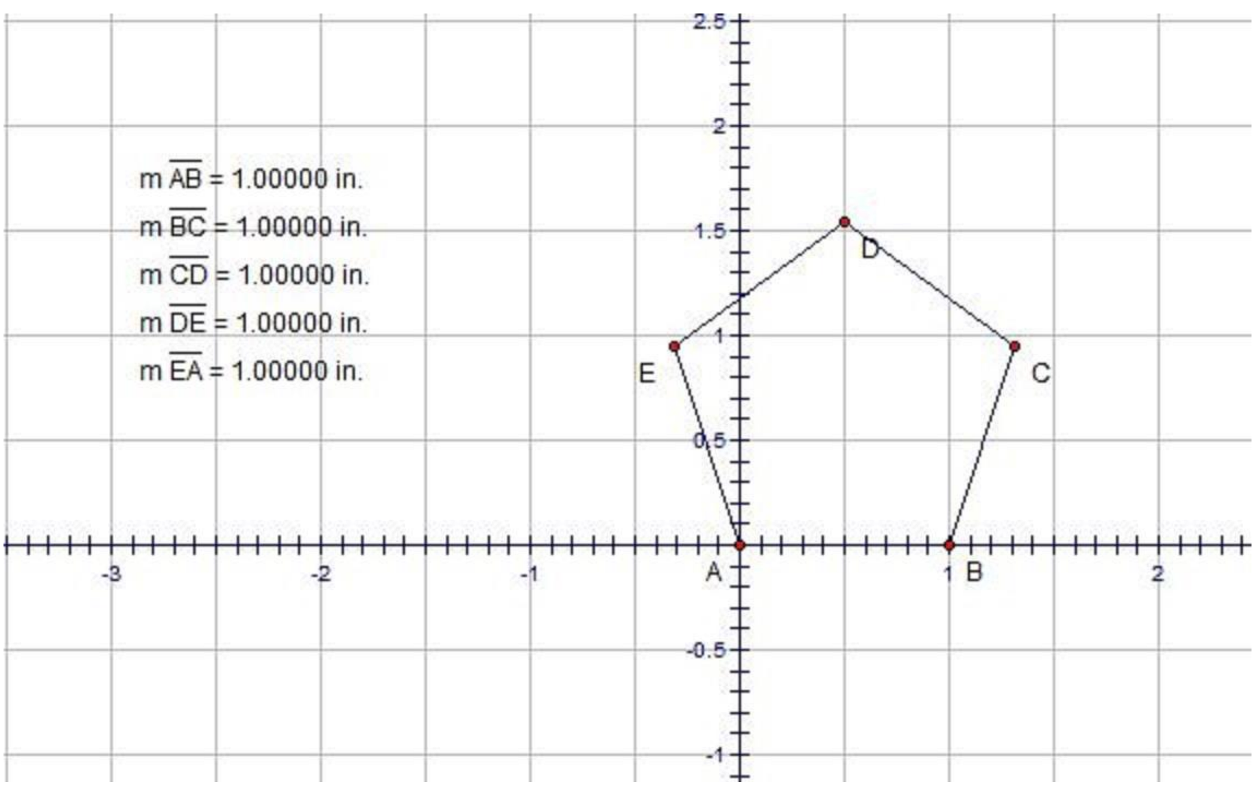


Image 1

Point A is also the first point of our quadrilateral. To obtain the second point of our quadrilateral, we can draw a unit circle with the center at B and another unit circle with the center at E. The circles will intersect at the points A and F as shown in Image 2. F is the second point of our quadrilateral. There are other methods of obtaining the point F. For example, F is the intersection of the line that bisects the angle BAE and the segment BD.

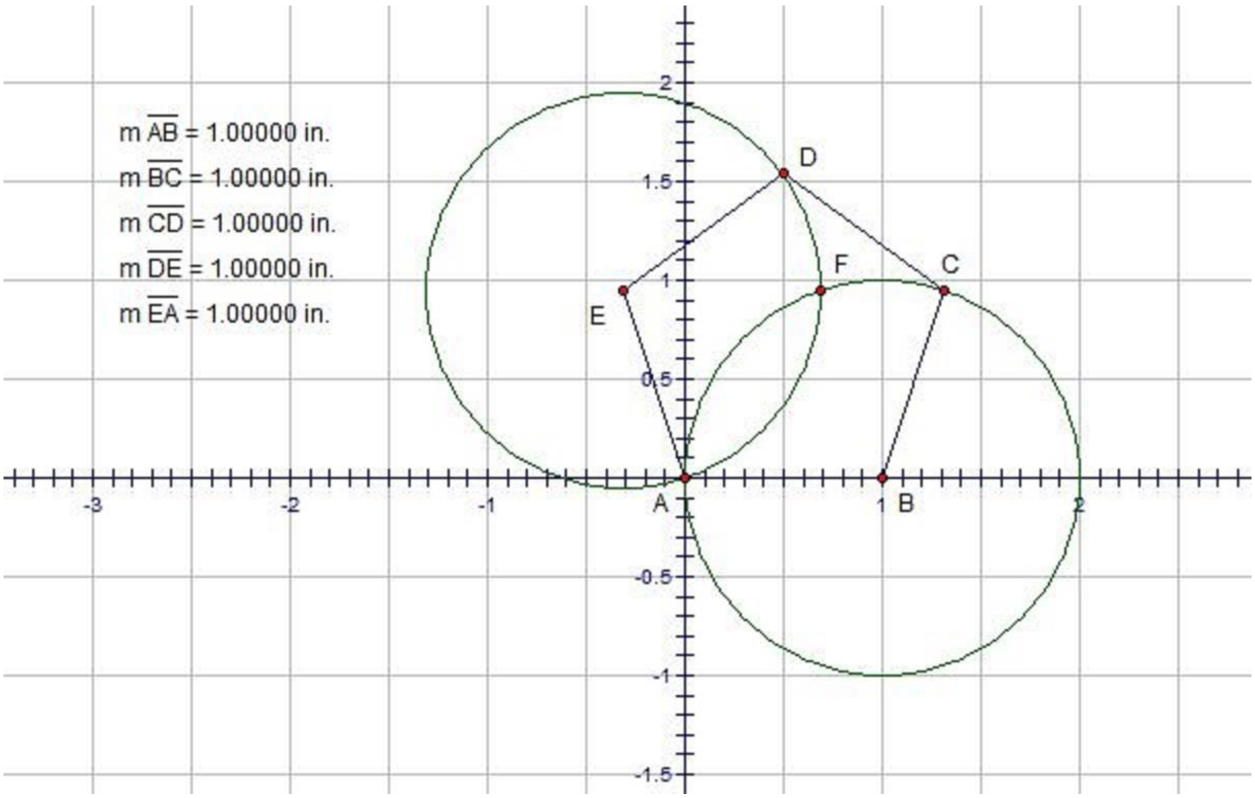


Image 2

To obtain the last 2 points, first we need to draw a line that extends though B and C. Now we need to construct a line that passes though F and it is perpendicular to the line though B and C. This line will intersect the unit circle with the center at B at F and G. G is our third point. Using the same method, we draw a line though A that intersects the circle at H. Image 3 shows the lines and the points.

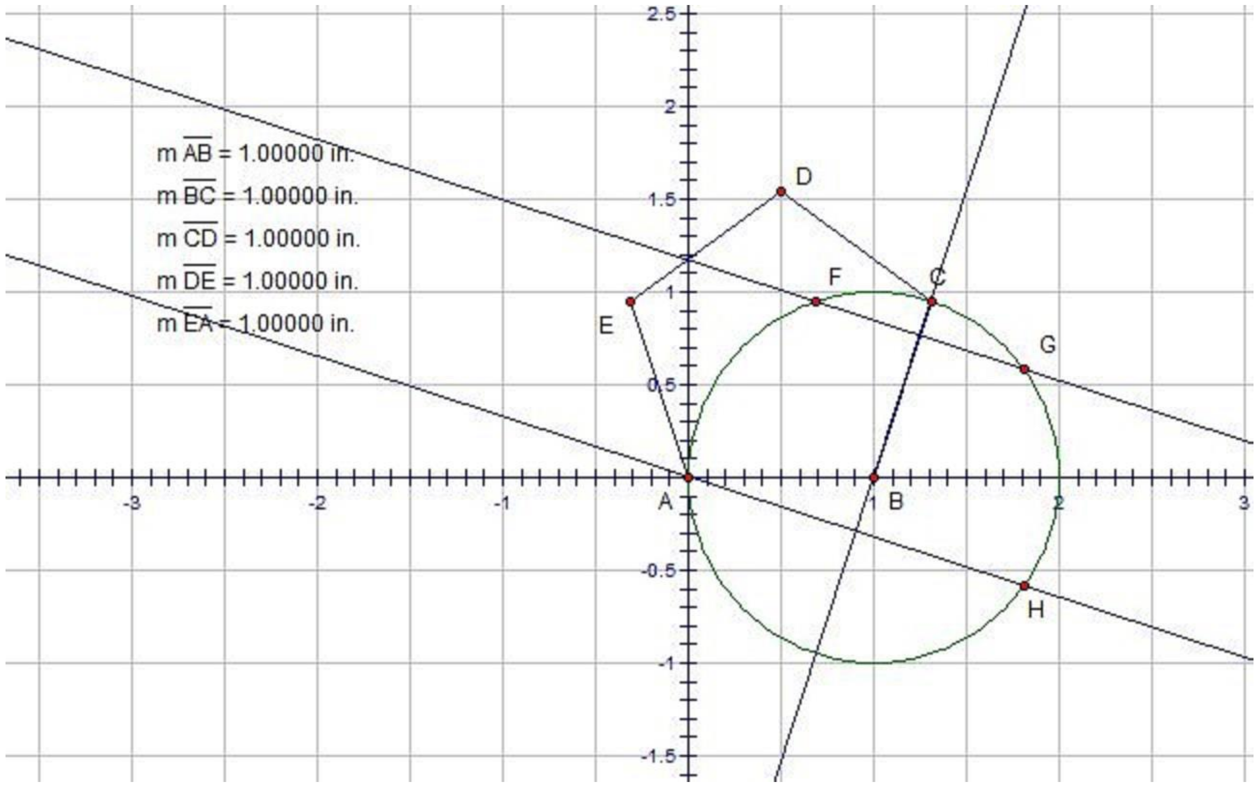
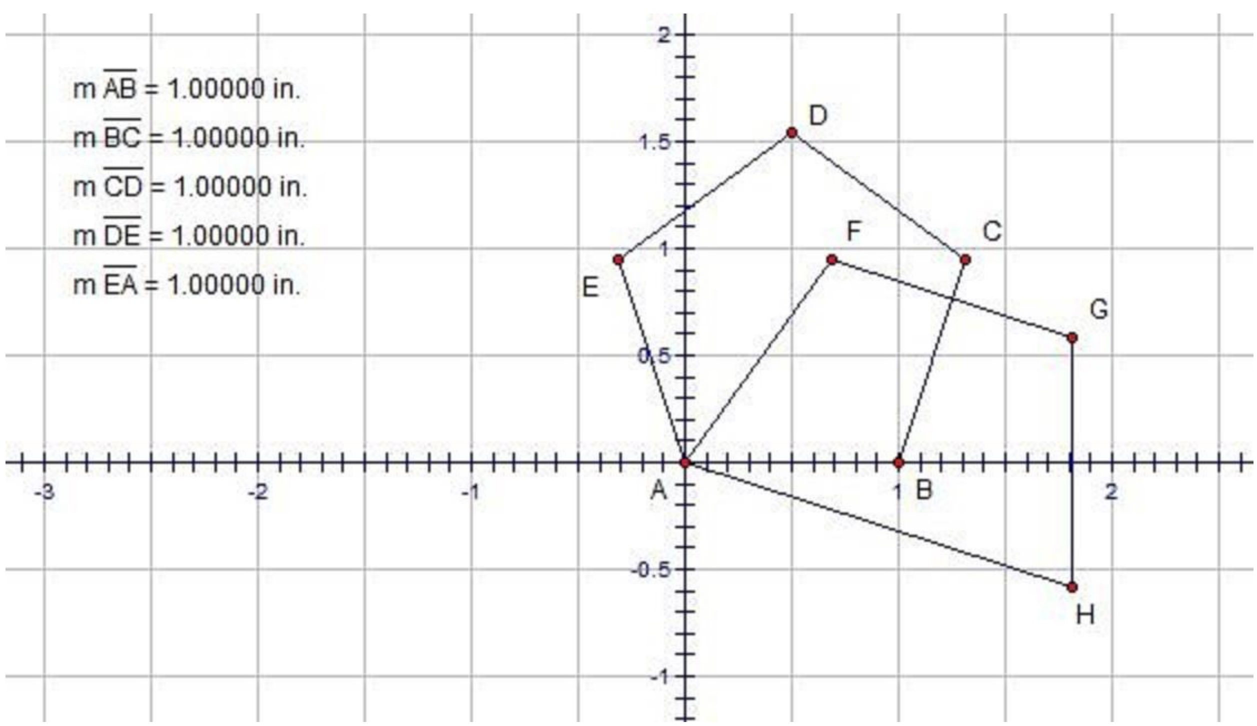


Image 3

Image 4 shows the pentagon and the quadrilateral.



Properties of the Quadrilateral

The first property of the quadrilateral that we can deduce is the fact that this quadrilateral is cyclic, since its vertices lie on the unit circle with the center at B. The sides of the quadrilateral are related to the golden ratio or φ≈ 1.618033. The length of AF=FG=GH= $\sqrt{4-\varphi^2} \approx 1.17557$. The length of AH= φ$\sqrt{4-\varphi^2} \approx 1.90211$.

AFGH also has the same area as the pentagon ABCDE. The area of a pentagon with unit length sides is about 1.720. Since AFGH is cyclic, we can use the Brahmagupta's formula to calculate the area. Brahmagupta's formula is similar to Heron's formula for triangles and it makes use of the semiperimeter to calculate the area. The semiperimeter s≈ 2.71441. The semiperimeter s matches the cube root of 20 to 5 decimal points, which I think is impressive. The cube root of 20 cannot be constructed with just a compass and the straightedge, but the length of s can be constructed with these 2 instruments. I can also point out that s matches Euler's number to 2 decimal points since e≈ 2.71828. This is not too impressive, but it is still interesting. It is also interesting to note that the golden ration seems to be always involved in interesting approximations. Another example is $(6/5)\varphi^2$, which approximates π to 3 decimal points (a construction that is not related to the golden ratio but approximates π to 4 decimal places is Kochanski's approximation).

I can also point out that the quadrilateral can be easily made into a regular pentagon. The line that passes though B and C, intersects the unit circle with the center at B at C and at another point that we can call I (see Image 3). Then the pentagon AFGHI, is a pentagon inscribed in the unit circle with the center at B. So, from a pentagon with unit sides we can obtain a pentagon inscribed in a unit circle.

Lill's method

At the beginning I mentioned the fact that a polygon can represent a polynomial equation when we use Lill's method. Actually, the initial pentagon can represent 2 polynomials. If we use the extension of Lill's method that I developed in one of my papers (see [1] from the sources section), the pentagon can represent a polynomial with complex coefficients. However, it would be more convenient if the pentagon would represent a polynomial with real coefficients. If we use the method presented at [2], the pentagon represents the polynomial $P(x)= x^4 + x^3 + x^2 + x +1$.

No matter what method we use, the points A, F, G and H gives us the roots of the polynomial P(x). The polynomials similar to P(x) are interesting, since the roots of these polynomials are roots of unity. In a future paper I will discuss Lill's method in relation to regular polygons.

An Interesting Patent

On a Google search I stumbled on an interesting patent [3]. The title of the patent is "Golden mean harmonized water and aqueous solutions". The patent is interesting since it talks about the golden mean (golden ratio), e, the cubic root of 20 and other mathematical numbers. These numbers seem to play a role in our physical reality, not just the mathematical reality.

Sources

[1] "Representing Polynomials with Complex Coefficients using Lill's Method"

[2] MEULENBELD, B., "*Note on LILL'S method of solution of numerical equations*". Proc. Kon. Ned Akad. V. Wetensch. 53, 464 (1950).

[3] https://patents.google.com/patent/US20120080305