Pell Infinite Series and the Negative Powers of the Silver Ratio

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Introduction

In this paper I want to present 2 infinite series formulas that are similar to the formulas presented in the paper '*Fibonacci Infinite Series and the Negative Powers of the Golden Ratio*". In the previous paper I showed 2 general Fibonacci infinite series that converge to the negative powers of the golden ratio (golden mean). In this paper I will show 2 infinite series formulas that have Pell numbers terms and converge to the negative powers of the silver ratio (silver mean or silver constant). The bigger implication is that similar formulas can be obtained for the negative powers of all other metallic means.

I also want to add that this paper will be informal, and I will try to be very brief. I will present the 2 general formulas, I will show a few concrete examples, then I will go briefly over the methods I used to obtain the formulas.

General Formulas

The Pell numbers belong to the integer sequence A000129 found at the OEIS website (The Online Encyclopedia of Integer Sequences). The Pell numbers are defined by the following recursive relation: P(k) = 2P(k-1) + P(k-2), with P(0) = 0 and P(1) = 1. The standard symbol for the silver ratio is δ_S. The 2 general formulas can be defined in the following manner:

for k = odd number

$$\delta_S^{-k} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} P(k)^2}{P(kn)P(kn+k)}$$

for k = even number

$$\delta_S^{-k} = \sum_{n=1}^{\infty} \frac{P(k)^2}{P(kn)P(kn+k)}$$

Some Examples

Now that we know the general formulas, we can go over some examples.

$$0.414213 \approx \delta_S^{-1} = \frac{1}{(1)(2)} - \frac{1}{(2)(5)} + \frac{1}{(5)(12)} - \frac{1}{(12)(29)} + \frac{1}{(29)(70)} \ldots + \frac{(-1)^{n+1}}{P(n)P(n+1)}$$

$$0.171572 \approx \delta_S^{-2} = \frac{2^2}{(2)(12)} + \frac{2^2}{(12)(70)} + \frac{2^2}{(70)(408)} + \frac{2^2}{(408)(2378)} + \frac{2^2}{(2378)(13860)} \ldots + \frac{2^2}{P(2n)P(2n+2)}$$

$$0.071067 \approx \delta_S^{-3} = \frac{5^2}{(5)(70)} - \frac{5^2}{(70)(985)} + \frac{5^2}{(985)(13860)} - \frac{5^2}{(13860)(195025)} + \frac{5^2}{(195025)(2744210)} \ldots + \frac{(-1)^{n+1}5^2}{P(3n)P(3n+3)}$$

$$0.029437 \approx \delta_S^{-4} = \frac{12^2}{(12)(408)} + \frac{12^2}{(408)(13860)} + \frac{12^2}{(13860)(470832)} + \frac{12^2}{(470832)(15994428)} + \frac{12^2}{(15994428)(543339720)} \dots + \frac{12^2}{P(4n)P(4n+4)}$$

Method

I want to go briefly over the method I used to obtain the 2 general formulas. To obtain the formulas I used Whittaker's formula for polynomial equations. Whittaker's formula can be used to find the smallest root of a polynomial (the root with the smallest absolute value). For more information about the Whittaker infinite series formula you can read [1] or my paper "Generating k-Pell Infinite Series Using the Whittaker Formula". To obtain the 2 infinite series formula I applied the Whittaker Formula on the polynomials $O_k(x) = -x^2 - A(k)\,x + 1$, for k= odd natural number, and $E_k(x) = x^2 - A(k)\,x + 1$, for k= even natural number. A(k) refers to the kth number that belongs to the Companion Pell sequence (A002203 from OEIS). The recursive relation for the Companion Pell numbers is: A(n) = 2A(n-1) + A(n-2), where A(0)= A(1) = 2. $O_k(x)$ has the roots δ_S^{-k} and $-\delta_S^{k}$, while $E_k(x)$ has the roots δ_S^{-k} and δ_S^{k}. In both cases, δ_S^{-k} gives the root with the smallest absolute value, so the application of Whittaker formula on the polynomials gives infinite series for δ_S^{-k}.

I should add that applying the Whittaker formula on a specific $O_k(x)$ or a $E_k(x)$ polynomial doesn't give you an infinite series that has the form similar to the infinite series that I listed in the "Some Examples" section. A few additional simple manipulations must be applied to get an infinite series similar to the ones presented in the previous section.

Final Notes

This paper and the paper '*Fibonacci Infinite Series and the Negative Powers of the Golden Ratio*" point to the fact that using similar methods we most likely can obtain similar infinite series formulas for all the other metallic means or metallic numbers. So, we probably can obtain 2 infinite series formulas that are more general and that can be used to obtain the negative powers of any metallic number.

References

[1] Whittaker E.T. and Robinson G., The Calculus of Observations, pp 120-123, 1924